Curvature for Hilbert modules, Kasparov modules and spectral triples

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Tori

Let us start with the usual flat metric on the non-commutative torus. Define a conformally related metric using $M = b\mathrm{Id}$, with $b \in \mathcal{A}_\theta$ positive and invertible. Then using a frame (ω_j) for the new metric, the curvature is

$$
\begin{aligned}
&R_M(\omega_k) \\
&= \sum_{r,l} \omega_r \otimes (1-\Psi)(\omega_l\omega_k)\big(2b^2\partial_r(b)b^{-1}\partial_l(b)b^{-1} - b^2\partial_l\partial_r(b)b^{-1}\big) \\
&+ \sum_{r,l} \omega_r \otimes (1-\Psi)(\omega_l\omega_r)\big(-2b\partial_l(b)b^{-1}\partial_k(b) + b\partial_l\partial_k(b)\big) \\
&- \sum_{r,l} \omega_r \otimes (1-\Psi)(\omega_r\omega_k)b\partial_l(b)\partial_l(b)b^{-1}.
\end{aligned}
$$

This is different to what is obtained from the heat kernel analogy. A careful analysis by Iochum and Masson has shown that for rational tori the second heat kernel coefficient computes the scalar curvature (divided by 6) PLUS a range of other terms coming from the non-scalar principal symbol.

We compute the scalar curvature as

$$\sum_{r,k}\langle R(\omega_r,\omega_k)\omega_k,\omega_r\rangle$$
$$= -(n-1)\sum_r \left(b\partial_r^2(b) + b^2\partial_r^2(b)b^{-1}\right) - n(n-1)\sum_r b\partial_r(b)\partial_r(b)b^{-1}$$
$$+ 2(n-1)\sum_r \left(b^2\partial_r(b)b^{-1}\partial_r(b)b^{-1} + b\partial_r(b)b^{-1}\partial_r(b)\right).$$

If $\theta = 0$ and $b = e^u$ we obtain the classical result

$$\sum_{r,k} \langle R(\omega_r, \omega_k)\omega_k, \omega_r\rangle$$
$$= -2(n-1)e^{2u}\Delta(u) - (n-1)(n-2)e^{2u}(\nabla(u))^2.$$

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Classically the curvature is an antisymmetric 2-form-valued endomorphism and so the diagonal entries are zero. In the noncomm case

$$(\omega_k | R(\omega_k)) = \sum_l (1-\Psi)(\omega_l\omega_k)\big(2b^2[\partial_k(b), b^{-1}\partial_l(b)b^{-1}] + b^2[b^{-1}, \partial_l\partial_r(b)]\big).$$

The curvature tensor for the Podleś sphere was computed using the frame coming from the columns of the matrix corepresentation t^1_{ij} of $SU_q(2)$. The metric is q-deformed, and while the junk is $(\sum \omega_j \omega_j^*)\mathcal{A}$, it is given by

$$\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \mathcal{A}.$$

We find

$$R = \sum_{i,r} (-1)^{1+i} |\omega_i\rangle \otimes \omega_i^* \wedge \omega_r \otimes \langle \omega_r|,$$

A Weitzenbock formula

Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D}) = (\mathcal{A}^\circ, L^2(X_\mathcal{A}, \phi), \mathcal{D})$. Nuisance.

Since $\mathcal{C}_\mathcal{D}(\mathcal{A}^\circ) \cong \mathcal{C}_\mathcal{D}(\mathcal{A})^\circ$ we can just consider right actions of $\mathcal{C}_\mathcal{D}(\mathcal{A})$. Use $c_R : L^2(X_\mathcal{A}, \phi) \otimes_\mathcal{A} \mathcal{C}_\mathcal{D}(\mathcal{A}) \to L^2(X_\mathcal{A}, \phi)$ to denote this action.

Given a right connection ∇^X on $X_\mathcal{A}$ and a left connection ∇^Ω on $\Omega^1_\mathcal{D}(\mathcal{A})$, define a connection Laplacian by

$$\Delta(x) = \Psi \circ (\nabla^X \otimes 1 + 1 \otimes \nabla^\Omega) \circ \nabla^X \in X \otimes_\mathcal{A} J^2_\mathcal{D}(\mathcal{A}).$$

Recall that in our main examples the junk is just $\mathcal{A}$ and so Δ is a map on $X_\mathcal{A}$.

Given the set-up above, we find a frame $(x_j)_{j=1}^N$ for the module $X_\mathcal{A}$. This is a (finite) set of generators such that for all $x \in X_\mathcal{A}$, $x = \sum_j x_j (x_j|x)_\mathcal{A}$.

Then $p = ((x_i|x_j)_\mathcal{A})$ is a projection and $X_\mathcal{A} \cong p\mathcal{A}^N$. Any (represented) connection is of the form

$$\nabla_\mathcal{D}(x) = \sum_j x_j \otimes [\mathcal{D}, (x_j|x)_\mathcal{A}] + x_j \otimes B^j_{kl}\omega^l (x_k|x)_\mathcal{A},$$

where (ω^l) is a frame for $\Omega^1_\mathcal{D}(\mathcal{A})$, $B^j_{kl} \in \mathcal{A}$.

When $J^2_\mathcal{D}(\mathcal{A}) = \mathcal{A}$ as in the main examples, we obtain a Weitzenbock type result.

Proposition

Suppose that $J^2_{\mathcal{D}}(\mathcal{A}) = \mathcal{A}$.
If ∇^{Ω} *is the Levi-Civita connection then* $\mathcal{D}^2 - \Delta$ *is* $\mathcal{A}$*-linear. In this case the difference is*

$$\mathcal{D}^2 - \Delta = \sum_{j,k} c_R\Big(x_k \otimes m(1-\Psi)\big([\mathcal{D},(x_k|x_j)_{\mathcal{A}}][\mathcal{D},(x_j|x_m)_{\mathcal{A}}]\big)(x_m|x)_{\mathcal{A}}\Big)$$
$$+ \sum_{k,j,l} c_R\Big(x_k \otimes \mathrm{d}_{\Psi}(B^{kl}_j\omega_l)(x_j|x)_{\mathcal{A}}\Big)$$
$$+ \sum_{k,l,m,p} c_R\Big(x_m \otimes m(1-\Psi)\big(B^{mp}_k\omega_p B^{kl}_j\omega_l\big)(x_j|x)_{\mathcal{A}}\Big)$$

Need to relate the curvature to the curvature of $\Omega^1_{\mathcal{D}}(\mathcal{A})$.

References

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Thanks for listening!