†

String^c Structures and Modular Invariants

Haibao Duan, **Fei Han** and Ruizhi Huang

National University of Singapore

Table of contents

Section 1

Backgrounds : (rational) genera, liftings of structure groups and Witten genus

Genera

- A **(rational) genus**
$$\varphi : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$$
from the (rational) cobordism ring of oriented manifolds, by definition, is a ring homomorphism such that $\varphi(1) = 1$.
- (Hirzebruch) Each genus uniquely corresponds to and is defined by a **characteristic power series**
$$Q(z) = 1 + a_2 z^2 + a_4 z^4 + \cdots ,$$
where "z" can be viewed as formal Chern root.

A-hat genus of spin manifolds : Atiyah-Singer index theorem

- $\hat{A}$**-genus**, by definition, corresponds to the characteristic power series

$$Q(z) = \frac{z/2}{\sinh(z/2)} = 1 - z^2/24 + 7z^4/5760 - 31z^6/967680 + \cdots ,$$

- (**Atiyah-Singer index theorem**) Let M be a $4k$-dimensional closed oriented smooth spin manifold (i.e. $\omega_2(M) = 0$), then

$$\mathrm{Ind} \not{D} \otimes E = \int_M \hat{A}(M) \cdot \mathrm{ch}(E \otimes \mathbb{C}),$$

  where $\not{D}$ is the Dirac operator and here twisted by the real vector bundle E over M.

- In particular, the $\hat{A}$-genus of a spin manifold is an integer.

Generalization of Atiyah-Singer (I) : From Spin to Spin^c

- A closed oriented manifold M is called *Spin^c* if there is an element $c \in H^2(M;\mathbb{Z})$ such that the mod 2 reduction $\rho_2(c) = \omega_2(M)$.
- To specify a Spin^c-structure (M, c) on M is equivalent to specify a pair (M, ξ), where

$$\mathbb{C} \to \xi \to M$$

  is the complex line bundle corresponding to c. May denote $M = (M, \xi, c)$.
- (Spin^c Atiyah-Singer) Let M be a $2k$-dimensional closed oriented smooth *Spin^c*, then

$$\mathrm{Ind} \not{D}^c = \int_M \hat{A}(M) \cdot e^{\frac{1}{2}c}.$$

- In particular, $\hat{A}(M) \cdot e^{\frac{1}{2}c}$ of a Spin^c manifold is an integer.

Generalization of Atiyah-Singer (II) : From Spin to String

- Consider the Whitehead tower of BSO

$$\cdots \to BString \xrightarrow{\frac{p_1}{2}} BSpin \xrightarrow{\omega_2} BSO.$$

- (**Witten (virtual) bundle**, 1988) For M^{4k},

$$\Theta(T_{\mathbb{C}}M) = \bigotimes_{m=1}^{+\infty} S_{q^{2m}}(\widetilde{T_{\mathbb{C}}M}),$$

where $q = e^{\pi i \tau}$ $(\tau \in \mathbb{H})$, $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$, $\widetilde{T_{\mathbb{C}}M} = T_{\mathbb{C}}M - \mathbb{C}^{4k}$, and the **total symmetric powers** of any bundle E

$$S_t(E) = 1 + tE + t^2 S^2(E) + \cdots .$$

- (Index theorem with **Witten form (genus)** ; Zagier, '86) If M^{4k} is *String*, then

$$\mathrm{Ind} \not{D}^L = \int_M \mathcal{W}(M) := \hat{A}(M) \cdot \mathrm{ch}(\Theta(T_{\mathbb{C}}M))$$

is an integral modular form of weight $2k$ over $SL(2, \mathbb{Z})$.

Recall definition of modular form

- Let Γ be a subgroup of $SL(2,\mathbb{Z})$.
- A **modular form** over Γ is a holomorphic function f on $\mathbb{H}$ such that

$$f(\frac{a\tau+b}{c\tau+d})=(c\tau+d)^k f(\tau)$$

for any $\tau\in\mathbb{H}$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma.$$

- In this talk, we are actually using its **Fourier expansion**.

Witten-Bott-Taubes-Liu's Rigidity Theorem

Witten-Bott-Taubes-Liu's Rigidity Theorem

Let X be a closed smooth connected manifold which admits a nontrivial S^1 action. Let P be an elliptic differential operator on X commuting with the S^1 action. Then the kernel and cokernel of P are finite dimensional representation of S^1. The equivariant index of P is the virtual character of S^1 defined by

$$\mathrm{Ind}(g, P) = \mathrm{tr}|_g \ker P - \mathrm{tr}|_g \mathrm{coker} P,$$

for $g \in S^1$. We call that P is *rigid* with respect to this circle action if $\mathrm{Ind}(g, P)$ is independent of g.

Theorem

The Witten operators $B_M \otimes \Theta_1(T_{\mathbb{C}}M), D \otimes \Theta_2(T_{\mathbb{C}}M)$ are rigid.

This implies in particular the *Rarita-Schwinger operator* $D \otimes T_{\mathbb{C}}M$ is rigid.

Liu's Vanishing Theorem

Liu's Vanishing Theorem (appended by Dessai)

Theorem

If M is a smooth string manifold and admits an nontrivial action of of S^3, then the Witten genus vanishes

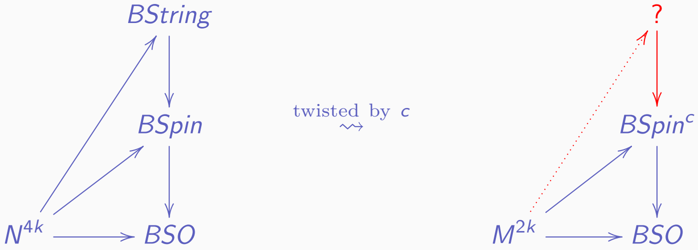
$$\Psi_W(M,\tau) = 0.$$

A profound development of the classical result by Atiyah-Hirzebruch :
If M is spin and admits a smooth S^1 action, then the A-hat genus vanishes :

$$\hat{A}(M) = 0.$$

Vast development of rigidity and vanishing theorems : Liu-Ma-Zhang family and foliation cases, Dessai spin^c case, Liu-Yu $\mathbb{Z}/k$ case, Mathai-H. noncompact case...

The Question : $? = String^c$



- (**Question :**) can we fill in the "?" in the diagram ? String^c-structures ? Twisted Witten genera ? Index theorem ? Applications ?

Partial answer

- A Spin^c (M, ξ, c) is **String^c of Chen-H-Zhang**
  - if $M = M^{4k}$ and $p_1(M) - 3c^2 = 0$ (rationally);
  - if $M = M^{4k+2}$ and $p_1(M) - c^2 = 0$ (rationally).
- (**Generalized Witten bundle**) For M^{4k},

$$\Theta(T_{\mathbb{C}}M, \xi_{\mathbb{R}} \otimes \mathbb{C}) = \Theta(T_{\mathbb{C}}M) \otimes \Lambda_{(+,0)}(\xi) \otimes \Lambda_{(+,1)}(\xi) \otimes \Lambda_{(-,1)}(\xi);$$

For M^{4k+2},

$$\Theta(T_{\mathbb{C}}M, \xi_{\mathbb{R}} \otimes \mathbb{C}) = \Theta(T_{\mathbb{C}}M) \otimes \Lambda_{(-,0)}(\xi),$$

where

$$\Lambda_{(\pm,0)}(\xi) = \bigotimes_{n=1}^{\infty} \Lambda_{\pm q^{2n}}(\widetilde{\xi_{\mathbb{R}} \otimes \mathbb{C}}), \quad \Lambda_{(\pm,1)}(\xi) = \bigotimes_{n=1}^{\infty} \Lambda_{\pm q^{2n-1}}(\widetilde{\xi_{\mathbb{R}} \otimes \mathbb{C}}),$$

and the **total exterior powers** of any bundle E

$$\Lambda_t(E) = 1 + tE + t^2\Lambda^2(E) + \cdots .$$

Partial answer

- (**Generalized Witten form (genus)**)

$$W_c(M) = \int_M \mathcal{W}_c(M) := \hat{A}(M)e^{\frac{1}{2}c}\mathrm{ch}(\Theta(T_{\mathbb{C}}M, \xi_{\mathbb{R}} \otimes \mathbb{C})).$$

- They are also indices of some so-called **generalized Witten operators**, and are integral modular forms of weight $2k$ over $SL(2;\mathbb{Z})$ when M is String^c in the sense of CHZ.

Our (further) answer of the question :

1. to give complete answer to the String^c-structures in the mentioned spirit ;
2. to construct String^c-groups ;
3. to exploit the algebraic topology of String^c structures ;
4. to construct generalized Witten genera which are integral modular forms (up to constants) for String^c-manifolds ;
5. to prove vanishing theorems analogous to those for String-manifolds and CHZ's String^c ;
6. apply vanishing theorem to almost complex manifolds and symplectic manifolds.

Section 2

Algebraic topology of String^c-structures and Generalized Witten Genus of Various Levels

Algebraic topology of $BSpin^c$

- By definition, the topological group $\mathbf{Spin^c(n)}$ is the central extension of $SO(n)$ by $U(1)$; alternatively, we have the principal bundle

$$Spin(n) \xrightarrow{i} Spin^c(n) \xrightarrow{\pi} S^1.$$

- For **free loop space** $LX = \mathrm{map}(S^1, X)$, we have the canonical fibration

$$\Omega X \xrightarrow{i} LX \xrightarrow{p} X,$$

  where p is the evaluation map.

- $LSpin$, $LSpin^c$, etc, are so-called **loop groups**.

Algebraic topology of $BSpin^c$, contiuned : Cohomology

$H^{i=?}(-;\mathbb{Z})$	1	2	3	4
$Spin(n)$	0	0	$\mathbb{Z}\{\mu_3\}$	0
$LSpin(n)$	0	$\mathbb{Z}\{x_2\}$	$\mathbb{Z}\{\mu_3\}$	$\mathbb{Z}\{x_2^2\}$
$Spin^c(n)$	$\mathbb{Z}\{s_1\}$	0	$\mathbb{Z}\{\mu_3\}$	$\mathbb{Z}\{s_1\mu_3\}$
$L_kSpin^c(n)$	$\mathbb{Z}\{s_1\}$	$\mathbb{Z}\{x_2\}$	$\mathbb{Z}\{s_1x_2\} \oplus \mathbb{Z}\{\mu_3\}$	$\mathbb{Z}\{s_1\mu_3\} \oplus \mathbb{Z}\{x_2^2\}$
$BSpin(n)$	0	0	0	$\mathbb{Z}\{q_4\}$
$BSpin^c(n)$	0	$\mathbb{Z}\{c_2\}$	0	$\mathbb{Z}\{c_2^2\} \oplus \mathbb{Z}\{q_4\}$
$BLSpin(n)$	0	0	$\mathbb{Z}\{\mu_3\}$	$\mathbb{Z}\{q_4\}$
$BLSpin^c(n)$	$\mathbb{Z}\{s_1\}$	$\mathbb{Z}\{c_2\}$	$\mathbb{Z}\{s_1c_2\} \oplus \mathbb{Z}\{\mu_3\}$	

- Remark : we have Spin^c-classes c_2^2, q_4 $(2q_4 + c_2^2 = p_1)$; $LSpin^c$-class s_1c_2, μ_3 ; etc.
- Duan (2018) has completely determined all the Spin^c-classes.
- General $LSpin$ ($LSpin^c$)-classes are still mysterious.

Algebraic topology of $BSpin^c$, contiued : "transgression"

- **The free evaluation map**

$$\mathrm{ev}: S^1 \times LX \to X$$

  is defined by $\mathrm{ev}((t,\lambda)) = \lambda(1)$.

- Define the **free (cohomology) suspension ("transgression")**

$$\nu : H^{n+1}(X) \to H^n(LX)$$

  by the formula $\mathrm{ev}^*(x) = 1 \otimes p^*(x) + s_1 \otimes \nu(x)$ for any $x \in H^{n+1}(X)$.

Lemma (Duan-H-Huang)

$\nu : H^4(BSpin^c(n);\mathbb{Z}) \to H^3(BLSpin^c(n);\mathbb{Z})$ satisfies

$$\nu(q_4) = \mu_3 - s_1 c_2, \quad \nu(c_2^2) = 2 s_1 c_2.$$

In particular,

$$\nu\left(\frac{p_1 - (2k+1)c^2}{2}\right) = \mu_3 - (2k+1) s_1 c_2, \quad \text{for any } k \in \mathbb{Z}.$$

A : general String^c structure via classifying spaces

From now on, let $(M^n, \xi, c = e(\xi))$ be a Spin^c-triple. For any $k \in \mathbb{Z}$,

- M is **level $2k+1$ (strong) String^c** if the characteristic class

$$\frac{p_1(M) - (2k+1)c^2}{2} = 0.$$

- M is **level $2k+1$ (weak) String^c** if the characteristic class

$$\mu_3(LM) - (2k+1)sc = 0,$$

  where μ_3, s is the "loop" of q_4 and u_2 respectively.

Remark : CHZ's String^c-manifolds rationally are level 3 when $M = M^{4m}$, and level 1 when $M = M^{4m+2}$.

B : constructing $String_k^c$-groups

- Embedding $Spin^c(n)$ to "larger" $Spin(N)$ groups. e.g., when $k<0$,

$$\begin{array}{ccc}
Spin^c(n) & \xrightarrow{\qquad\qquad\lambda_{2k+1}\qquad\qquad} & Spin(n-4k-2) \\
\Big\downarrow \rho & & \Big\downarrow p \\
SO(n)\times S^1 \xrightarrow{\mathrm{id}_{SO(n)}\times\Delta_{-2k-1}} & SO(n)\times\underbrace{S^1\times\cdots\times S^1}_{-2k-1} \xhookrightarrow{\chi_{-2k-1}} & SO(n-4k-2),
\end{array}$$

- Constructing $String_k^c(n)$ by pull-back of topological groups

$$\begin{array}{ccc}
String_k^c(n) & \xrightarrow{\gamma_{2k+1}} & String(N) \\
\Big\downarrow j_k & \ulcorner & \Big\downarrow j \\
Spin^c(n) & \xrightarrow{\lambda_{2k+1}} & Spin(N),
\end{array}$$

- Hence, $String_k^c$-group is an extension of $Spin^c(n)$ by $K(\mathbb{Z},2)$.
- Stolz-Teichner (2004) defined $PU(A)$ as a model of $K(\mathbb{Z},2)$.
- Nikolaus-Sachse-Wockel (2013) Kac-Moody group model of String.

From “B” to “A”

For each k, we already have the extension

$$\{1\} \to PU(A) \to String_k^c(n) \xrightarrow{j_k} Spin^c(n) \to \{1\}.$$

$$\begin{array}{ccc} & & BString_k^c(n) \\ & \nearrow & \downarrow \\ M^n & \xrightarrow[g]{} & BSpin^c(n) \\ & & \downarrow {\scriptstyle \frac{p_1-(2k+1)c_2^2}{2}} \\ & & K(\mathbb{Z},4) \end{array} \quad \overset{\text{transgress to}}{\rightsquigarrow} \quad \begin{array}{ccc} & & BLString_k^c(n) \\ & \nearrow & \downarrow \\ LM^n & \xrightarrow[Lg]{} & BLSpin^c(n) \\ & & \downarrow {\scriptstyle u_3-(2k+1)s_1c_2} \\ & & LK(\mathbb{Z},4) \end{array}$$

Nonloop/Strong Case — Loop/Weak Case

C : structural theorem of (strong) *String^c*-manifolds

Theorem String^c (Duan-H-Huang)

- A equivalence :

M admits a strong String^c-structure	$\Longleftrightarrow$	The stable spin principal bundle associated to $M \oplus \xi^{\oplus(-2k-1)}$ admits a string structure for some $k \in \mathbb{Z}$

- If M is $(2k+1)$-level String^c,

The distinct level k strong String^c-structures on M	$\overset{1-1}{\longleftrightarrow}$	The elements in the image of $\rho^* : H^3(M) \to H^3(S(\xi))$

where $\rho : S(\xi) \to M$ is the circle bundle of ξ.

C : structural theorem of (weak) String^c-manifolds

LTheorem String^c (Duan-H-Huang)

- A equivalence :

M admits a weak String^c-structure	$\Longleftrightarrow$	The structural group of the LSpin^c-principal bundle of LM can be lifted to $\widehat{LSpin^c}(n)$ in a reasonable way

- If M is $(2k+1)$-level weak String^c,

The distinct level k weak String^c-structures on M	$\overset{1-1}{\longleftrightarrow}$	The elements in the image of $(L\rho)^* : H^2(LM) \to H^2(LS(\xi))$

Remark : Here $\widehat{LSpin^c}(n)$ is a universal central extension extension of $LSpin^c(n)$ by $U(1)$

C : summary

	Strong/Nonloop point of view		Weak/ Loop point of view	
G-Structure	obstruction characteristic class	counting (parameterized by)	structural group lifting to	counting (parameterized by)
SO	$\omega_1(M)$	$H^0(M;\mathbb{Z}/2)$		
$Spin$	$\omega_2(M)$	$H^1(M;\mathbb{Z}/2)$	$L_0SO(n)$	$H^0(LM;\mathbb{Z}/2)$
String	$\frac{p_1(M)}{2}$	$H^3(M;\mathbb{Z})$	$L\widehat{Spin}(n)$	$H^2(LM;\mathbb{Z})$
Spin^c	$W_3(M)$	$H^1(M;\mathbb{Z}/2) \oplus 2H^2(M;\mathbb{Z})$		
$String_k^c$	$\frac{p_1(M)-(2k+1)c^2}{2}$	$\mathrm{Im}\left(\rho^* : H^3(M) \to H^3(S(\xi))\right)$	$L\widehat{Spin^c}(n)$	$\mathrm{Im}((L\rho)^*)$

Remark : “Strong” implies “Weak”, while the converse holds only under

D : generalized Witten genera

From "D" to "E", let (M, ξ, c) be a level $2k+1$ (strong) String^c-manifold such that $2k+1 > 0$.

Let $\vec{a} = (a_1, a_2, \cdots, a_r) \in \mathbb{Z}^r, \vec{b} = (b_1, b_2, \cdots, b_s) \in \mathbb{Z}^s$ be two vectors.

- If $M = M^{4m}$, suppose
$$3||\vec{a}||^2 + ||\vec{b}||^2 = 2k - 2;$$
- if $M = M^{4m+2}$, suppose
$$3||\vec{a}||^2 + ||\vec{b}||^2 = 2k.$$

Recall

$$\Lambda_{(\pm,0)}(E) = \bigotimes_{n=1}^{\infty} \Lambda_{\pm q^{2n}}(\widetilde{E_{\mathbb{R}} \otimes \mathbb{C}}), \quad \Lambda_{(\pm,1)}(E) = \bigotimes_{n=1}^{\infty} \Lambda_{\pm q^{2n-1}}(\widetilde{E_{\mathbb{R}} \otimes \mathbb{C}}).$$

D : generalized Witten genera, continued

Generalized Witten bundles

- Define $(\vec{a}, \vec{b})$-indexed virtual bundle

$$\Upsilon_{\vec{a},\vec{b}}(T_{\mathbb{C}}M, \xi_{\mathbb{R}} \otimes \mathbb{C}) := \Theta(T_{\mathbb{C}}M) \otimes$$
$$\bigotimes_{i=1}^{r} \left(\Lambda_{(+,0)}(\xi^{\otimes a_i}) \otimes \Lambda_{(+,1)}(\xi^{\otimes a_i}) \otimes \Lambda_{(-,1)}(\xi^{\otimes a_i})\right) \bigotimes_{j=1}^{s} \Lambda_{(-,0)}(\xi^{\otimes b_j}).$$

- For M^{4k},

$$\Theta_{\vec{a},\vec{b}}(T_{\mathbb{C}}M, \xi_{\mathbb{R}}\otimes\mathbb{C}) := \Upsilon_{\vec{a},\vec{b}}(T_{\mathbb{C}}M, \xi_{\mathbb{R}}\otimes\mathbb{C})\otimes\Lambda_{(+,0)}(\xi)\otimes\Lambda_{(+,1)}(\xi)\otimes\Lambda_{(-,1)}(\xi);$$

- For M^{4k+2},

$$\Theta_{\vec{a},\vec{b}}(T_{\mathbb{C}}M, \xi_{\mathbb{R}} \otimes \mathbb{C}) := \Upsilon_{\vec{a},\vec{b}}(T_{\mathbb{C}}M, \xi_{\mathbb{R}} \otimes \mathbb{C}) \otimes \Lambda_{(-,0)}(\xi).$$

D : generalized Witten genera, continued

- **Generalized Witten forms**

$$\mathcal{W}^c_{2k+1,\vec{a},\vec{b}}(M) := \widehat{A}(M)e^{\frac{c}{2}}\prod_{j=1}^{r}\cosh\left(\frac{a_jc}{2}\right)\prod_{j=1}^{s}\sinh\left(\frac{b_jc}{2}\right)$$
$$\cdot\mathrm{ch}\left(\Theta_{\vec{a},\vec{b}}(T_{\mathbb{C}}M,\xi_{\mathbb{R}}\otimes\mathbb{C})\right);$$

- **Generalized Witten genera**

$$W^c_{2k+1,\vec{a},\vec{b}}(M) = \int_M \mathcal{W}^c_{2k+1,\vec{a},\vec{b}}(M).$$

Modularity Theorem

The generalized Witten genera are integral modular forms of weight $2m$ over $SL(2,\mathbb{Z})$ up to a scaler $1/2^{r+s}$ which only depends on k.

Remark : For $(M^{4m}, k=1)$ and $(M^{4m+2}, k=0)$, the generalized Witten forms and genera, and their integrality and modularity reduce to those of CHZ respectively.

E : Liu's type vanishing theorem

Theorem (Duan-H-Huang)

Let (M, ξ, c) be a level $2k+1$ (strong) String^c-manifold such that $2k+1 > 0$. If M admits an effective positive action of a simply connected compact Lie group that can be lifted to the Spin^c structure, then

$$W^c_{2k+1;\vec{a},\vec{b}}(M) = 0.$$

Positive condition of action inspired by Liu

Under the condition of the theorem, we have for G-equivariant characteristic classes

$$p_1(M)_G - (2k+1)c_1(\xi)^2_G = \alpha \cdot \pi^* q,$$

where $\pi : M \times_G EG \to BG$, and $q \in H^4(BG)$ is the canonical generator. The G-action is **positive** if $\alpha > 0$.

E : Remark on positive condition

$$p_1(M)_G - (2k+1)c_1(\xi)_G^2 = \alpha \cdot \pi^* q, \quad (\alpha > 0).$$

An example :

On $M = \mathbb{C}P^{2n}$, consider the Spin^c-structure $(\mathbb{C}P^{2n}, c(\xi))$ determined by the stable almost complex structure

$$T\mathbb{C}P^{2n} \oplus \mathbb{R}^2 \cong \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-1),$$

where there are $n+1$ many $\mathcal{O}(1)$, n-many $\mathcal{O}(-1)$. Then
$c(\xi) = c_1(J) = x$ and $\mathbb{C}P^{2n}$ is String^c of level $2n+1$.
Now the linear action of $SU(2n+1)$ on $\mathbb{C}P^{2n}$ preserves J, which is positive (indeed $\alpha = 1$).

Section 3

Applications of generalized vanishing theorem

G : applications to almost complex (a.c.) manifolds

Theorem (Duan-H-Huang)

Let $(M, J, c = c_1(J))$ be a level $2k+1$ (strong) String^c almost complex manifold such that $2k+1 > 0$. If M admits an effective positive action of a simply connected compact Lie group that preserves the almost complex structure J, then

$$W^c_{2k+1;\vec{a},\vec{b}}(M) = 0.$$

G : applications to almost complex (a.c.) manifolds : a special case

Theorem (Duan-H-Huang)

Let (M^{2n}, J) be a closed almost complex manifold, which is level $2k+1$ String^c. Then if

- $2k - n \geq 18$, and
- $c_1^n(J) \neq 0$ rationally,

M does not admit a positive effective action of any simply connected compact Lie group preserving J.

H : applications to homotopy projective spaces

A **complex homotopy projective space** M^{2n}, by definition, is a manifold $M^{2n} \simeq \mathbb{C}P^n$.

- The **Petrie conjecture** (1972) claims that if S^1 acts effectively on a homotopy complex projective space X^{2n}, then the total Pontryagin class $p(X^{2n}) = p(\mathbb{C}P^n)$.
- The conjecture was proved for X^{2n} with $n \leq 4$, and by Hatorri (1978) when X^{2n} admits an S^1-invariant stable almost complex structure with $c_1 = (n+1)x$.
- (Hatorri 1978) when $c_1 = kx$ with $|k| > n+1$, X^{2n} admits no S^1 action preserving J.

H : applications to homotopy projective spaces

Dessai proved that :
If X^{4n} is a homotopy CP^{2n} and $p_1 > (2n+1)x^2$, then X^{4n} does not support nontrivial smooth S^3 action.

Using our vanishing theorem, we can reprove this result.

Just observe that X^{4n} is String^c of level $(2n+1) + 24\rho(X)$ and apply the above corollary.

I : applications to prequantizable symplectic manifolds

Theorem (Duan-H-Huang)

Let $(M, \omega, c = [\omega])$ be a level $2k+1$ (strong) String^c prequanizable symplectic manifold such that $2k+1 > 0$. Suppose M admits an effective symplectic action of a simply connected compact Lie group. If the action is Hamiltonian and positive, then

$$W^c_{2k+1;\vec{a},\vec{b}}(M) = 0.$$

Thank you very much !