Analysis on Trees and Buildings

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October 2nd, 2019
Analysis on Manifolds, Adelaide

Australian Government
Australian Research Council

CARMA

Differential geometry without differentiation

Manifolds to graphs

real Lie group	$\longrightarrow$	totally disconnected, locally compact group
symmetric space	$\longrightarrow$	vertex-transitive, locally finite graph

The comparison is made in two stages.

I. Lie groups over t.d.l.c. fields

II. General t.d.l.c. groups

I. Lie groups over t.d.l.c. fields

Proposition

Every topological field is either connected or totally disconnected.

Theorem (van Dantzig, Jacobson, Pontyagin 1930s)

1. *Every connected locally compact field is either $\mathbb{R}$ or $\mathbb{C}$.*
2. *Every non-discrete, totally disconnected, locally compact (t.d.l.c.) field is either:*
   2.1 $\mathbb{Q}_p$, *with p prime, or a finite extension (characteristic 0); or*
   2.2 $\mathbb{k}_q((t))$ *with $\mathbb{k}_q$ a finite field (positive characteristic).*

Totally disconnected locally compact fields

Definition

- The field of *p-adic numbers*, with p prime is

$$\mathbb{Q}_p := \left\{ \sum_{n \geq N} a_n p^n \mid a_n \in \{0, \dots, p-1\} \right\},$$

with p ‘carried’ in addition and multiplication.

- The field of *formal Laurent series* over the finite field $\Bbbk_q$ is

$$\Bbbk_q(\!(t)\!) := \left\{ \sum_{n \geq N} a_n t^n \mid a_n \in \Bbbk_q \right\},$$

with coordinatewise addition and convolution multiplication.

$\mathfrak{O}$ denotes the *ring of integers*, either $\mathbb{Z}_p$ or $\Bbbk_q[\![t]\!]$.

Buildings are analogues of symmetric spaces

Manifolds to trees and buildings

real Lie group	$\longrightarrow$	Lie group over t.d.l.c. field
symmetric space	$\longrightarrow$	regular tree or building

Buildings as homogeneous spaces

Let $G = PGL_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{Q}_p$ or $\mathbb{k}_q((t))$ and $U = PGL_n(\mathfrak{O})$. Then $U \leq G$ is compact and open, and G/U is a homogeneous space which has the discrete topology and is countable.

Definition

A *lattice* in $\mathbb{K}^n$ is an $\mathfrak{O}$-submodule of the form $\mathfrak{O} \cdot v_1 \oplus \cdots \oplus \mathfrak{O} \cdot v_n$ with $\{v_1, \ldots, v_n\}$ a basis for $\mathbb{K}^n$.
Lattices L_1, L_2 are *equivalent* if there $\lambda \in \mathbb{K}$ with $L_2 = \lambda L_1$.
Denote the set of equivalence classes of lattices by Λ.

Proposition

- *Equivalence of lattices is invariant under $GL_n(\mathbb{K})$.*
- *$PGL_n(\mathbb{K}) \curvearrowright \Lambda$ and is transitive.*
- *The stabiliser of $[L_0]$, with $L_0 = \mathfrak{O}^n$ is equal to U.*
- *G/U may be identified with Λ.*

Lattices as vertices of a graph

Let π be the *uniformiser* in $\mathbb{K}$, that is $\pi = p$ or $\pi = t$. Define

$$\mathscr{E} = \left\{([L_1],[L_2]) \in \Lambda^2 \mid L_1 > L_2 > \pi L_1\right\}.$$

Proposition

- *$(\Lambda, \mathscr{E})$ is a graph (undirected).*
- *The action of $PGL_n(\mathbb{K})$ preserves the edge relation.*
- *Suppose that $L_0 > L > \pi L_0$, i.e., $[L]$ is a neighbour of $[L_0]$. Then $L/\pi L_0 = [x_1 : \cdots : x_n] \in P^{n-1}(\mathfrak{O}/\pi\mathfrak{O}) = P^{n-1}(\mathbb{k}_q)$.*

Note: $\mathfrak{O}/\pi\mathfrak{O}$ is the *residue field* of $\mathbb{K}$ and is isomorphic to $\mathbb{k}_q$. If $(\xi_1, \ldots, \xi_n) \in L \setminus L_0$, then $(\xi_1 + \pi\mathfrak{O}, \ldots, \xi_1 + \pi\mathfrak{O}) \in \mathbb{k}_q^n \setminus \{0\}$ and are the homogeneous coordinates of a point in $P^{n-1}(\mathbb{k}_q)$.

When $n = 2$ the building is a regular tree

Remarks

- When $n = 2$, the building $(\Lambda, \mathscr{E})$ is a $(q + 1)$-regular tree.
- $PGL_2(\mathbb{K})$ is the 'rank 1' case. In general, the building of $PGL_n(\mathbb{K})$ is an $(n - 1)$-simplicial complex and the above describes only the 1-skeleton of this simplicial complex.
- The set of ends of tree may be identified with the projective line $P^1(\mathbb{K})$.

Groups acting on regular trees

Many other groups act on regular trees.

- The automorphism, or isometry, group, $\mathrm{Isom}(T_n)$, of the n-regular tree is a t.d.l.c. group.
- The free group $\mathbb{F}_k$ acts on its Cayley graph, which is T_{2k}.
- Amalgamated free products and HNN-extensions of groups act on trees, by Bass-Serre theory, *e.g.*, $\underbrace{C_2 * \cdots * C_2}_{n} \curvearrowright T_n$

A group, G, has Serre's *Property FA* if every action of G on a tree has a fixed point. A finitely generated group G has FA if and only if:
(i) G is not an amalgamated free product; and
(ii) G does not have $\mathbb{Z}$ as a quotient group;

Analysis on trees

The analogy between the symmetric space $PSL_2(\mathbb{R})/SO_2(\mathbb{R})$ and the homogeneous space $PGL_2(\mathbb{K})/PGL_2(\mathfrak{O})$, with $\mathbb{K}$ a t.d.l.c. field, inspired parallels being drawn between harmonic analysis of $PSL_2(\mathbb{R})$ and of groups acting on trees.

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Buildings in general

- The construction of buildings for $PGL_n(\mathbb{K})$ extends to all semisimple Lie groups over t.d.l.c. fields.
- A Kac-Moody group over a finite field may be represented as acting on a building, and the closure of the group in the permutation topology is a t.d.l.c. group.
- J. Tits has shown how to construct a building from a group which has a *BN-pair*.
- J. Tits has given the abstract definition: a building is a combinatorial geometry build from *chambers* and *apartments* satisfying certain axioms.

Symmetric spaces and buildings

Theorem (Caprace & Monod)

Let X be a geodesically complete proper CAT(0)-space. Suppose that the stabilizer of every point at infinity acts cocompactly on X. Then X is isometric to a product of symmetric spaces, Euclidean buildings and Bass-Serre trees.

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Analysis on buildings

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II. General t.d.l.c. groups

Isometry groups of trees and buildings are totally disconnected and locally compact. The structure of the groups is encoded in the geometry of the trees and buildings.

Manifolds to graphs

real Lie group	$\longrightarrow$	totally disconnected, locally compact group
symmetric space	$\longrightarrow$	Cayley-Abels graph

Question: What is structure of general t.d.l.c. groups? How much more general than buildings are their ‘geometries’?
Answer: Much more general but how much more is not known.

Compact open subgroups

Theorem (van Dantzig, 1930's)

- *Let G be a t.d.l.c. group and $\mathcal{N}$ be a neighbourhood of the identity. Then there is a compact open subgroup $U \subset \mathcal{N}$.*
- *Every compact t.d.l.c. group is profinite.*

Notation

$\mathcal{LCO}(G) = \{\,\text{compact open subgroups of } G\,\}$.

- These are also known as *0-dimensional groups* because their inductive and topological dimensions equal 0.
- No topological invariants (such as dimension) distinguish between the groups.

Cayley-Abels graphs

Definition

Let G be a t.d.l.c. group with compact generating set $X = X^{-1}$ and let $U \in \mathcal{LCO}(G)$. Suppose that $X = UXU$.
Put $\mathscr{V} = G/U$ and $\mathscr{E} = \{(gU, gxU) \in \mathscr{V}^2 \mid x \in X\}$.
Then $\Gamma(G; U, X) := (\mathscr{V}, \mathscr{E})$ is a *Cayley-Abels graph* for G.

Proposition

$\Gamma(G; U, X)$ is a locally finite graph and $G \curvearrowright \Gamma(G; U, X)$.

The 1-skeleton of Bruhat-Tits building of $PGL_n(\mathbb{K})$ can be realised as a Cayley-Abels graph.
The tree T_n can be recovered a Cayley-Abels graph of $\mathrm{Isom}(T_n)$.

The scale and minimizing subgroups

Definition

Let $\alpha \in \mathrm{End}(G)$. The *scale of* α is the positive integer

$$s(\alpha) := \min \left\{ [\alpha(U) : \alpha(U) \cap U] : U \in \mathcal{LCO}(G) \right\}.$$

The compact open subgroup U of G is *minimizing for* α if the minimum is attained at U.

The structure of minimizing subgroups for automorphisms

Theorem

Let $\alpha \in Aut(G)$ and $U \leq G$ be compact and open. Define

$$U_+ := \bigcap_{k \geq 0} \alpha^k(U) \text{ and } U_- := \bigcap_{k \geq 0} \alpha^{-k}(U).$$

Then U is minimizing for α if and only if

TA $U = U_+ U_-$; *and*

TB $U_{++} := \bigcup_{k \geq 0} \alpha^k(U_+)$ *is closed.*

If U is minimizing, then $s(\alpha) = [\alpha(U_+) : U_+]$.

A similar characterisation applies for endomorphisms.

The tree representation theorem

For $x \in G$, write α_x for the inner automorphism $y \mapsto xyx^{-1}$ and $s(x) = s(\alpha_x)$.

Theorem (U. Baumgartner & W.)

Let $a \in G$ and suppose that $s(x) > 1$. Suppose that U is tidy for x. Then $U_{++} \rtimes \langle x \rangle$ is a closed subgroup of G. Moreover, there is a homomorphism

$$\rho : U_{++} \rtimes \langle x \rangle \to Isom(T_{s(x)+1})$$

such that:

- *$\rho(U_{++} \rtimes \langle x \rangle)$ fixes an end, ∞, of $T_{s(x)+1}$;*
- *$\rho(x)$ is a hyperbolic element of $Isom(T_{s(x)+1})$ and translates a geodesic, $(v_n)_{n \in \mathbb{Z}}$, towards ∞.*
- *$\rho(U_{++})$ consists of elliptic elements and $\rho(U_+)$ is the stabiliser of the vertex v_0.*

Tidy subgroups for commuting automorphisms

Commuting matrices may be simultaneously triangularized.

Theorem

Let $\mathcal{H}$ be a finitely generated abelian group of automorphisms of the t.d.l.c. group G. Then there is a compact open subgroup U of G that is tidy for every α in $\mathcal{H}$.

The commutator of triangular matrices is unipotent.

Theorem

Let α and β be automorphisms of the t.d.l.c. group G and suppose that there is a compact open subgroup U tidy for every automorphism in $\langle \alpha, \beta \rangle$. Then

$$\alpha\beta\alpha^{-1}\beta^{-1}(U) = U.$$

Tidy subgroups as a canonical form

Definition

1. A subgroup $\mathcal{H} \leq \mathrm{Aut}(G)$ is *flat* if there is $U \in \mathcal{LCO}(G)$ that is tidy for every $\alpha \in \mathcal{H}$.
2. The *uniscalar* subgroup of $\mathcal{H}$ is

$$\mathcal{H}_1 = \left\{\alpha \in \mathcal{H} \mid s(\alpha) = 1 = s(\alpha^{-1})\right\}$$

$\mathcal{H}_1$ is a subgroup because $\alpha \in \mathcal{H}_1$ if and only if $\alpha(U) = U$ for any, and hence all, subgroups tidy for $\mathcal{H}$.

Tidy subgroups as a canonical form

Theorem

Let $\mathcal{H}$ be a finitely generated flat group of automorphisms of the t.d.l.c. group G and suppose that U is tidy for $\mathcal{H}$. Then $\mathcal{H}_1 \triangleleft \mathcal{H}$ and there is $r \in \mathbb{N}$ such that

$$\mathcal{H}/\mathcal{H}_1 \cong \mathbb{Z}^r.$$

1. *There is $k \in \mathbb{N}$ such that*

   $$U = U_0 U_1 \dots U_k,$$

   where for every $\alpha \in \mathcal{H}$: $\alpha(U_0) = U_0$ and for every $j \in \{1, 2, \dots, k\}$ either $\alpha(U_j) \leq U_j$ or $\alpha(U_j) \geq U_j$.

2. For each $j \in \{1, 2, \dots, k\}$ there is a homomorphism $\rho_j : \mathcal{H} \to \mathbb{Z}$ and a positive integer s_j such that

$$[\alpha(U_j) : U_j] = s_j^{\rho_j(\alpha)}.$$

3. For each $j \in \{1, 2, \dots, k\}$,

$$\widetilde{U}_j := \bigcup_{\alpha \in \mathcal{H}} \alpha(U_j)$$

   is a closed subgroup of G.
4. The natural numbers r and k, the homomorphisms $\rho_j : \mathcal{H} \to \mathbb{Z}$ and positive integers s_j are independent of the subgroup U tidy for α.

Tidy subgroups as a canonical form

- The numbers $s_j^{\rho_j(\alpha)}$ are analogues of absolute values of eigenvalues for α.
- The subgroups $\bigcup_{\alpha\in\mathcal{H}} \alpha(U_j)$ are the analogues of common eigenspaces for the automorphisms in $\mathcal{H}$.

Example

$G = SL(n, \mathbb{Q}_p)$, $H = \{$ diagonal matrices in $GL(n, \mathbb{Q}_p)\}$ and $\alpha_h(x) = hxh^{-1}$. Then:

- $r = n - 1$;
- $k = n(n - 1)$;
- ρ_j are roots of H; and
- $\widetilde{U}_j$ are root subgroups of G.

Geometry and flat-rank

Theorem (U. Baumgartner, R. Möller & W.)

Let G be a compactly generated t.d.l.c. group and suppose that G is hyperbolic, i.e., that the Cayley-Abels graph $\Gamma(G; U, X)$ *is hyperbolic. The any flat subgroup of G has flat-rank at most equal to* 1.

Theorem (U. Baumgartner, B. Rémy & W.)

Let G be a t.d.l.c. group and suppose that G acts properly and co-compactly on the building Γ. *Then the flat-rank of G, that is, the maximal flat-rank of any flat subgroup, is equal to the geometric rank of* Γ.

Thank you for your attention