Equivariant Index Theory on Non-Compact Manifolds

Analysis on Manifolds

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The Heat Equation and Dirac Operators

The Heat Partial Differential Equation on $\mathbb{R}^n$

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The **heat kernel** on $\mathbb{R}^n$ is given by

$$K_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t}$$

for $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.

Properties of the Heat Kernel

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$$\frac{\partial u}{\partial t} + \Delta u = f,$$

the heat kernel

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Define $u(t, x) := \int_{\mathbb{R}^n} K_t(x, y) f(y) \, \mathrm{d}y$ then $u(t, x)$ solves the heat equation and also has the property $\lim_{t \to 0} u(t, x) = f(x)$.

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Definition

The **heat operator** $e^{-t\Delta}$ is $(e^{-t\Delta} f)(x) := \int_{\mathbb{R}^n} K_t(x,y) f(y)\,\mathrm{d}y$

Note this definition is not standard. Normally we use functional calculus.

The Laplacian Operator on a Manifold

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Now we can ask a question, "does there exist a first order linear differential operator that squares to a Laplacian?"

Dirac Operators on Manifolds

Definition

Let S be a complex vector bundle over a Riemannian manifold M. Then a **Dirac operator** $D : \Gamma^\infty(S) \to \Gamma^\infty(S)$ is a first order differential operator such that $D^2 = \Delta$ where Δ is a generalised Laplacian.

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Why Dirac operators over just the normal Laplacian?

- Dirac operators are first order
- Dirac operators occur naturally in mathematics and physics
- Dirac operators tell you about the geometry of M

The Heat Partial Differential Equation and the Heat Kernel on Manifolds

Definition

Let S be a complex vector bundle over a Riemannian manifold M and let D be a Dirac operator on S. Furthermore, let $s \in \Gamma^\infty(S)$ depending on some time parameter t. Then a **heat equation** on M is

$$\frac{\partial s}{\partial t} + D^2 s = f \tag{1}$$

We call this a generalised heat equation.

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Like before we look for a solution.

Theorem

There exists a unique $K_t(x, y)$ *such that* $u(t, x) = \int_M K_t(x, y) f(y)\, \mathrm{d}y$ *is a solution to a generalised heat equation and* $\lim_{t \to 0} u(t, x) = f(x)$.

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Well consider the homogeneous equation $\frac{\partial u}{\partial t} + D^2 u = 0 \Leftrightarrow \frac{\partial u}{\partial t} = -D^2 u$ and let $u(t,x) = e^{-tD^2}s_0$. Then

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial t}(e^{-tD^2}s_0) \text{``}=\text{''} -D^2 e^{-tD^2}s_0 = -D^2 u(t,x)$$

for some initial condition s_0 and thus it satisfies the heat equation.

Integral Kernels

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Definition

Let S_1 and S_2 be vector bundles over M. A bounded operator $T : L^2(S_1) \to L^2(S_2)$ is called a **smoothing operator** if there is a smooth kernel $K_T(m, m')$ such that

$$(Ts)(m) = \int_M K_T(m, m')s(m')\,\mathrm{d}m'$$

Index Theory

Dirac operators are Fredholm on Compact Manifolds

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We have the following theorem.

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Let S be a complex vector bundle over a Riemannian manifold M and let $D : \Gamma^\infty(S) \to \Gamma^\infty(S)$ be a Dirac operator. Suppose also that $S = S^+ \oplus S^-$, and further that $D(\Gamma^\infty(S^\pm)) \subset \Gamma^\infty(S^\mp)$. Define $D^\pm := D|_{S^\pm}$. Then if M is compact, $D^+ : \Gamma^\infty(S^+) \to \Gamma^\infty(S^-)$ is Fredholm.

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In fact the above theorem holds in more general settings, more specifically it holds for any elliptic differential operator of which the Dirac operators are a subset.

Atiyah-Singer Index Theorem

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Theorem (Atiyah–Singer, 1963)

Let D be a Dirac operator on a compact Riemannian manifold M. Then

$$\operatorname{ind}(D) = \int_M AS(D)$$

The left-hand side is analytic, while the right-hand side is purely defined in terms of the geometry of M.

Quick proof of the Atiyah-Singer Index theorem

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$$\mathrm{Tr}(e^{-tD^-D^+}) - \mathrm{Tr}(e^{-tD^+D^-}) = \int_M \left(\mathrm{tr}(K_t^+(m,m)) - \mathrm{tr}(K_t^-(m,m))\right) \mathrm{d}m$$

is independent of t.

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By McKean-Singer, as $t \to \infty$

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On the other hand, by heat kernel asymptotics at $t \to 0$ we have

$$\int_M \left(\mathrm{tr}(K_t^+(m,m)) - \mathrm{tr}(K_t^-(m,m))\right) \mathrm{d}m \to \int_M AS(D) \quad \square$$

Equivariant case and Φ-Index

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Definition

Let $\Phi : S \to S$, and $\varphi : M \to M$, (where S is a vector bundle over a compact Riemannian manifold M) be diffeomorphisms such that for all $m \in M$ we have

(1) $\Phi(S_m) \subset S_{\varphi(m)}$

(2) $\Phi_m : S_m \to S_{\varphi(m)}$ linear

and further that $\Phi \circ D = D \circ \Phi$. Then the **Φ-index of D** is $\text{ind}_\Phi(D) = \text{Tr}(\Phi \text{ on } \ker D^+) - \text{Tr}(\Phi \text{ on } \ker D^-)$.

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Note that if we take $\varphi = \mathrm{id}_M$ and $\Phi = \mathrm{id}_S$ then $\mathrm{ind}_\Phi(D) = \mathrm{ind}(D)$.

The Atiyah-Segal-Singer Index Theorem

Theorem (Atiyah–Segal–Singer, 1968)

Suppose Φ and φ correspond to an element of a compact, connected group acting on M and S. Let D be a Dirac operator on a compact Riemannian manifold M. Then

$$\mathrm{ind}_\Phi(D) = \int_{M^\varphi} AS_\Phi(D)$$

where M^φ is the fixed point set of M under φ.

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Proof: $\mathrm{ind}_\Phi(D) = \mathrm{Tr}(\Phi e^{tD^-D^+}) - \mathrm{Tr}(\Phi e^{-tD^+D^-})$

$$= \int_M \left(\mathrm{tr}(\Phi_{\varphi^{-1}(m)} K_t^+(\varphi^{-1}(m), m)) - \mathrm{tr}(\Phi_{\varphi^{-1}(m)} K_t^-(\varphi^{-1}(m), m))\right) \mathrm{d}m$$

$$= \int_{M^\varphi} AS_\Phi(D)$$

where the second equality follows at $t \to \infty$ and the last equality follows as $t \to 0$ with a localisation argument.

John Roe's Paper

Compact Manifolds versus Non-Compact Manifolds

Compact Manifolds versus Non-Compact Manifolds

Compact Manifold	Non-Compact Manifold
All elliptic operators, and in particular Dirac operators, are Fredholm	Elliptic operators are not necessarily Fredholm
There is a canonical Sobolev norm	Norms on Sobolev spaces depend on choices
The trace of a smooth kernel operator is well-defined	When calculating the trace of a smooth kernel operator we run into convergence issues as the integral might diverge

Riemannian Manifolds of Bounded Geometry

Definition

Let M be a complete Riemannian manifold. Then M is said to have **bounded geometry** if

- M has positive injectivity radius, and
- The curvature tensor of M is uniformly bounded, as are all its covariant derivatives.

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Example: $\mathbb{R}^n$

We can also talk about vector bundles of bounded geometry - just ignore the first assumption above to get the definition.

Sobolev Spaces

We need to make precise what spaces we are working with.

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Definition

Let M be a Riemannian manifold of bounded geometry and S a complex vector bundle of bounded geometry with a Dirac operator D. Let k be a nonnegative integer. The **Sobolev space** $W^k(S)$ is the completion of $\Gamma_c^\infty(S)$ in the norm

$$\|s\|_k = \left(\|s\|_{L^2}^2 + \|Ds\|_{L^2}^2 + \ldots + \left\| D^k s \right\|_{L^2}^2 \right)^{1/2}.$$

Furthermore, define $W^{-k}(S) := (W^k(S))^*$.

Construction of the set of uniform operators

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Definition

Let M, S be as above. Then $\mathscr{U}_{-\infty}(S) := \{T : \Gamma_c^\infty(S) \to \Gamma^\infty(S) \mid T$ is linear and for all $k, \ell \in \mathbb{Z}$ T extends continuously to an operator $W^k(S) \to W^{k-\ell}(S)\}$.

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Theorem

Let $T \in \mathscr{U}_{-\infty}(S)$ and $s \in \Gamma_c^\infty(S)$. Then T is smoothing operator with a bounded smoothing kernel $K_T(m, m')$ such that

$$Ts(m) = \int K_T(m, m')s(m')\,\mathrm{d}m'$$

Moreover, there is a function $v = v(r)$ that tends to 0 as $r \to \infty$ such that

$$\int_{M \setminus B(m,r)} \left|K_T(m, m')\right|^2 \mathrm{d}m' < v(r), \quad \int_{M \setminus B(m,r)} \left|K_T(m', m)\right|^2 \mathrm{d}m' < v(r)$$

Regular Exhaustions

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Definition

Let $K \subset M$ and define $\mathrm{Pen}^+(K, r) := \overline{\cup\{B(m, r) \mid m \in K\}}$. Furthermore, define $\mathrm{Pen}^-(K, r) := \overline{M \setminus \mathrm{Pen}^+(M \setminus K, r)}$.

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Definition

Let (M_i) be a sequence of increasing compact sets where $M_i \subset M$ for all i. Then (M_i) is said to be a **regular exhaustion** of M if for each $r \geq 0$ the quotient

$$\mathrm{Vol}(\mathrm{Pen}^+(M_i,r))/\,\mathrm{Vol}(\mathrm{Pen}^-(M_i,r))$$

tends to 1 as $i \to \infty$.

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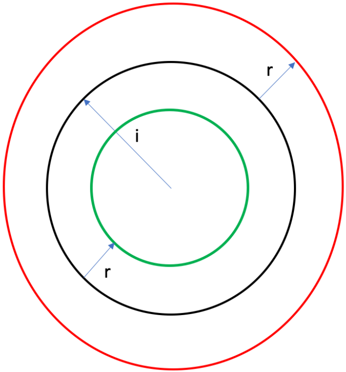

Figure: Black circle $= B(0, i) = M_i$, Red circle $= \text{Pen}^+(M_i, r)$, and Green circle $= \text{Pen}^-(M_i, r)$

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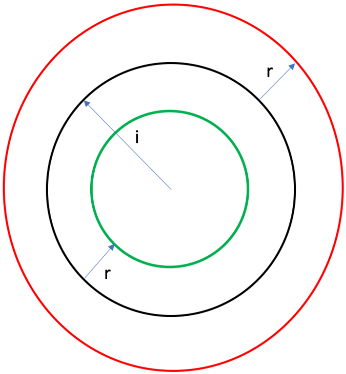

Figure: Black circle $= B(0, i) = M_i$, Red circle $= \mathrm{Pen}^+(M_i, r)$, and Green circle $= \mathrm{Pen}^-(M_i, r)$

$$\mathrm{Vol}(\mathrm{Pen}^+(M_i, r)) = C(i + r)^n$$

$$\mathrm{Vol}(\mathrm{Pen}^-(M_i, r)) = C(i - r)^n$$

So,

$$\frac{\mathrm{Vol}(\mathrm{Pen}^+(M_i, r))}{\mathrm{Vol}(\mathrm{Pen}^-(M_i, r))} = \frac{C(i + r)^n}{C(i - r)^n} \to 1$$

as $i \to \infty$.

Linear functionals Associated to Regular Exhaustions

Definition

Let M be a Riemannian manifold of dimension n. Let $\Omega^n_\beta(M)$ be the set of bounded n-forms on M. An element $I \in \Omega^n_\beta(M)^*$ is said to be **associated** to a regular exhaustion (M_i) of M if for each bounded n-form α,

$$\liminf_{i\to\infty} \left| \langle \alpha, I \rangle - \frac{1}{\operatorname{Vol} M_i} \int_{M_i} \alpha \right| = 0.$$

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Theorem

There exists functionals associated to every regular exhaustion.

The Functional τ

Let (M_i) be a regular exhaustion of M a Riemannian manifold with bounded geometry and let I be associated to the regular exhaustion. Let $T \in \mathscr{U}_{-\infty}(S)$ with bounded smoothing kernel K_T. The n-form $\alpha : m \to \mathrm{tr}(\mathrm{K_T}(m, m))$ is therefore bounded. Define $\tau(T) := \langle I, \alpha \rangle$ so that τ is a linear functional on $\mathscr{U}_{-\infty}(S)$.

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A functional T is said to be a **trace** on an algebra A if for all $a, b \in A$ we have $T(ab) = T(ba)$.

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The proof uses regular exhaustions and also relies on the construction of the uniform operators and the corresponding theorem which used the assumption of bounded geometry.

Roe's Index theorem

In light of the above we make the following definition.

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$\mathrm{ind}_\tau(D) := \tau(e^{-tD^-D^+}) - \tau(e^{-tD^+D^-})$, where everything is as above.

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We have $\mathrm{ind}_\tau(D) = I(AS(D))$

As an intuitive proof, we have the following set of equalities

$$\begin{aligned}\mathrm{ind}_\tau(D) &= \tau(e^{-tD^-D^+}) - \tau(e^{-tD^+D^-}) \\ &= \lim_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)} \int_{M_i} \left(\mathrm{tr}(K_t^+(m,m)) - \mathrm{tr}(K_t^-(m,m))\right) \mathrm{d}m \\ &= I(AS(D))\end{aligned}$$

where, like before, the second equality follows at $t \to \infty$ and the last follows as $t \to 0$.

New work

Generalising τ to τ_Φ

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Definition

Let T_t be an element of an algebra of paths of operators in $\mathscr{U}_{-\infty}$ with smoothing kernel K_t and Φ, and φ be as in the equivariant case, with Φ bounded. Let I be associated to (U_i) a regular exhaustion of a tubular neighbourhood of the fixed point set M^φ, and define $\alpha_\Phi(T) : m \mapsto \Phi_{\varphi^{-1}(m)} K_t(\varphi^{-1}(m), m)$ and given $\alpha_\Phi(T)$ define $\tau_\Phi(T) := \langle I, \alpha_\Phi(T) \rangle$.

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Lemma

$\tau_\Phi(t)$ is a trace on the algebra of paths of operators $\mathscr{U}_{-\infty}(S)$ as $t \to 0$.

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Continuing with our supposition, make the obvious choice of definition

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Conjecture

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Note this is just an adjusted Roe's theorem adapted for our purposes. However, the proof that $\tau_\Phi(S)$ is a trace on $\mathscr{U}_{-\infty}$ is permissible after adjusting the proof given by Roe but localising to the fixed point set is complicated.

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Finally, the resulting integral might depend on which tubular neighbourhood U of M^{φ} we pick when actually we want this to be independent of this choice.

Solutions

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To simplify matters we are first considering group actions with compact fixed point sets.