Restriction of eigenfunctions to sparse sets on manifolds

Joint work with Suresh Eswarathasan

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Plan

- Background and context
- Statement of the main results
- Overview of the proof

Plan

- **Background and context**
  - What is eigenfunction restriction?
  - A bit of history
  - Setting up the problem
- Statement of the main results
- Overview of the proof

Some basics

- (M, g) : a compact Riemannian manifold without boundary, dim d.

- Δ : Laplace-Beltrami operator associated with the metric g.

- $\{-\lambda_k^2 : k \geq 0\}$: sequence of distinct eigenvalues of Δ,

$$0 = \lambda_0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \longrightarrow \infty.$$

- $\mathbb{E}_\lambda$: eigenspace of $-\lambda^2$, i.e., $\Delta = -\lambda^2$ on $\mathbb{E}_\lambda$, $\quad \lambda = \lambda_0, \lambda_1, \cdots$.

It is known that

$$\dim(\mathbb{E}_{\lambda_k}) = m_k < \infty, \qquad L^2(M) = \bigoplus_k \mathbb{E}_{\lambda_k}.$$

Some examples

Example 1: The d-torus $\mathbb{T}^d$

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \qquad \lambda_k^2 = k, \qquad k = 0, 1, 2, \cdots ,$$

$$\mathbb{E}_{\lambda_k} = \text{span}\{e(n \cdot x),\ e(-n \cdot x) : n \in \mathbb{Z}^d,\ |n|^2 = k\}$$

Example 2: The Euclidean d-sphere $\mathbb{S}^d$

$$\lambda_k^2 = k(k + d - 1), \qquad k = 0, 1, 2, \cdots ,$$

$$\mathbb{E}_{\lambda_k} = \{\text{spherical harmonics of degree } k\}.$$

The general questions

High energy asymptotics

For eigenfunctions $\varphi_\lambda \in \mathbb{E}_\lambda$,

- What is the large scale behaviour of φ_λ as $\lambda \to \infty$?
- How do the eigenfunctions φ_λ "grow" or "concentrate"?
- For example, what can one say about
  - how large φ_λ can be?
  - the set where φ_λ is large?
  - the set where it vanishes?

These questions are inherently qualitative, but their quantitative reformulations are many!

I. Semiclassical Wigner measures

- Fix an ordered orthonormal basis $\{\psi_{k,j} : 1 \le j \le m_k\}$ of $\mathbb{E}_{\lambda_k}$.
- Get an ordered ONB $\bigcup_k \{\psi_{k,j} : 1 \le j \le m_k\}$ of $L^2(M)$.
- Associate to the ONB a sequence of distributions on T^*M: each wave function $\psi_{k,j}$ defines a probability measure

$$|\psi_{k,j}(x)|^2 \, d\mathrm{Vol}(x),$$

which can be lifted to a probability measure $dU_{k,j}$ on T^*M.

Questions of interest

1. What are all the weak* limit points of $\{dU_{k,j}\}$?
2. Is a given limiting measure "easily accessible"?

Semiclassical invariant measures (ctd)

Typically, one expects most of the eigenfunctions to equidistribute, i.e.,

$$\int_E |\varphi_\lambda(x)|^2 d\mathrm{Vol}(x) \approx \frac{\mathrm{Vol}(E)}{\mathrm{Vol}(M)} \text{ for most large } \lambda,$$

but can there be exceptional subsequences leading to other invariant measures?

- Shnirelman (1974)
- Colin de Verdiere (1985)
- Zelditch (1987, $\cdots$)
- Helffer-Martinez-Robert (1987)
- Sarnak (1995, $\cdots$)
- Anantharaman (2004, $\cdots$)
- Lindenstrauss (2006, $\cdots$)
- Hassell (2010) $\cdots$

II. Lebesgue norms of eigenfunctions

Relevant questions

- (Linear estimates) For $2 \leq p \leq \infty$, find $\delta(p)$ such that

$$\sup_{\varphi_\lambda \in \mathbb{E}_\lambda} \frac{||\varphi_\lambda||_p}{||\varphi_\lambda||_2} = O((1+\lambda)^{\delta(p)}).$$

- (Bilinear and multilinear versions) For example, find $\kappa(p)$ such that

$$||\varphi_\lambda \varphi_\mu||_{p/2} \leq C(1+\lambda)^{\kappa(p)} ||\varphi_\lambda||_2 ||\varphi_\mu||_2,$$

for $\varphi_\lambda \in \mathbb{E}_\lambda, \varphi_\mu \in \mathbb{E}_\mu$.

- Sogge (1988)
- Sogge and Zelditch (2002)
- Burq, Gerard and Tzvetkov (2004, 2005, $\cdots$)

Lebesgue norms (ctd) : Linear estimates

- Sogge (1988) : general compact Riemannian manifold (M, g)

Theorem (Sogge 1988) for $d = 2$

$$\frac{||\varphi_\lambda||_p}{||\varphi_\lambda||_2} = O((1+\lambda)^{\delta(p)}), \quad 2 \leq p \leq \infty,$$

where

$$\delta(p) = \begin{cases} \dfrac{1}{2} - \dfrac{2}{p} & \text{for } 6 \leq p \leq \infty, \\ \dfrac{1}{4} - \dfrac{1}{2p} & \text{for } 2 \leq p \leq 6. \end{cases}$$

- $\exists$ manifolds (M, g) for which estimates are sharp, e.g. $M = \mathbb{S}^2$.
- Connections with Stein-Tomas L^2 restriction theorem.

III. Growth of restricted Lebesgue norms

- $\Sigma \subseteq M$: A smooth embedded submanifold of dimension n, equipped with canonical measure endowed by the metric g.

Yet another question

- How well-behaved is φ_λ restricted to Σ?
- In particular, study growth of Lebesgue norms of φ_λ on Σ. Look for optimal exponents $\alpha(p, \Sigma)$ such that

$$||\varphi_\lambda||_{L^p(\Sigma)} \leq C(1+\lambda)^{\alpha(p)}||\varphi_\lambda||_{L^2(M)}.$$

- Reznikov (2004)
- Burq, Gerard, Tzvetkov (2007)
- Hu (2009)

Restricted eigenfunction growth (ctd)

A representative result (BGT 2007, Hu 2009)

For $d = 2$, and $\gamma : [0, 1] \to M$ a smooth curve, there exists a constant C such that

$$||\varphi_\lambda||_{L^p(\gamma)} \le C(1 + \lambda)^{\alpha_p} ||\varphi_\lambda||_{L^2(M)},$$

where

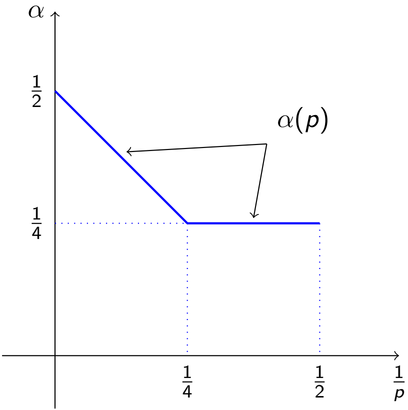
$$\alpha(p) = \left\{ \begin{array}{ll} \dfrac{1}{2} - \dfrac{1}{p} & \text{if } 4 \le p \le \infty, \\ \dfrac{1}{4} & \text{if } 2 \le p \le 4. \end{array} \right\}$$

Sharp for $M = \mathbb{S}^2$;

- any curve γ for $4 \le p \le \infty$.
- geodesic curve γ for $2 \le p < 4$.

- Versions available for general Σ and n.

Eigenfunction restriction from surfaces to curves



Optimal growth of Lebesgue norms

Comparisons

- An improvement over the Sobolev trace theorem: e.g., for $p = 2$,
  - the trace theorem gives a bound $(1 + \lambda)^{\frac{1}{2}}$,
  - BGT-H gives $(1 + \lambda)^{\frac{1}{4}}$.
  - indicates an improvement of the trace theorem when taken from the subclass of Laplace-Beltrami eigenfunctions.

- Partial averaging effect on the Weyl pointwise bound: for $p = \infty$,

$$||\varphi_\lambda||_{L^\infty(M)} \leq C(1 + \lambda)^{\frac{1}{2}} ||\varphi_\lambda||_{L^2(M)}.$$

  - the Weyl law is sharp for $M = \mathbb{S}^2$.
  - can view BGT-H as a result of averaging $|\varphi_\lambda|$ along a curve γ.
  - gain of $\lambda^{\frac{1}{4}}$ if averaged say in $L^4(\gamma)$; compare with $||\varphi_\lambda||_{L^4(M)} = O(\lambda^{\frac{1}{8}})$.

Sharpness, or lack thereof ...

BGT-H bound is sharp in general (as in $M = \mathbb{S}^2$), but

- need not be sharp for all γ:
  - estimate improves if γ has non-vanishing geodesic curvature in any M.
  - for example, $\alpha(2)$ becomes $\frac{1}{6}$ instead of $\frac{1}{4}$.
- not necessarily optimal even for every M and some γ:
  - for example, $M = \mathbb{T}^2$.
  - $L^2 \to L^\infty$ Weyl bound improves from $\lambda^{\frac{1}{2}}$ to λ^ϵ, any $\epsilon > 0$.
  - results in an improvement on BGT-H.

Formulation of the problem

A question of restriction

- Suppose $E \subseteq M$ is a set,
  - of specified size, in terms of Hausdorff dimension
  - equipped with some structure (to be determined)
  - supports a probability measure μ,

  but not necessarily a submanifold.
- How do Lebesgue norm estimates change when the Laplace-Beltrami eigenfunctions are restricted to E?
- Specifically, we look for estimates of the form

$$||\varphi_\lambda||_{L^p(E,\mu)} \leq C(1+\lambda)^{\alpha(p)}||\varphi_\lambda||_{L^2(M)}.$$

Plan

- Background and context
- **Statement of the main result**
  - Discussion of sharpness
  - The probabilistic setup
- Overview of the proof

Statement of the main result

- a submanifold $\Sigma \subseteq M$ of dimension $n \leq d$,
- small $0 \leq \epsilon < 1$,

$$p_0 := \frac{4n(1-\epsilon)}{d-1}.$$

Theorem (Eswarathasan-P 2019)

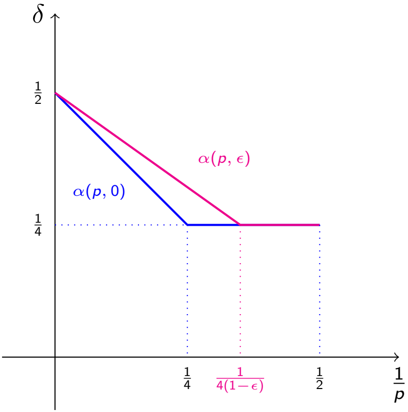
There exists a probability measure space $(\Omega, \mathcal{X}, \mathbb{P}^*)$:

For $\mathbb{P}^*$-a.e. $\omega \in \Omega$, $\exists$ a Cantor-type set $E_\omega \subsetneq \Sigma$ of Hausdorff dimension $n(1-\epsilon)$ and a constant $C_\omega > 0$ such that

$$||\varphi_\lambda||_{L^p(E_\omega)} \leq C_\omega (1+\lambda)^{\alpha_p} \Phi_p(\lambda) ||\varphi_\lambda||_{L^2(M)}, \text{ with}$$

$$\alpha_p := \begin{cases} \frac{d-1}{4} & \text{if } 2 \leq p \leq p_0, \\ \frac{d-1}{2} - \frac{n(1-\epsilon)}{p} & \text{if } p_0 \leq p \leq \infty. \end{cases}$$

Eigenfunction restriction to fractals: $d = 2$, $n = 1$



Growth of Lebesgue norms

A discussion of the growth rates

- Our exponent $\alpha_p = \frac{d-1}{2} - \frac{n(1-\epsilon)}{p}$ for large p is consistent with:
  - $\delta(p, d) = \frac{d-1}{2} - \frac{d}{p}$ when $\Sigma = M$ (Sogge 1988).
  - $\delta(p, n) = \frac{d-1}{2} - \frac{n}{p}$ for submanifold $\Sigma \subseteq M$ of dim n (BGT 2007).

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- Many sets E with the same restriction estimate as BGT-H:
  - if $d = n = 2, \epsilon = 1/2$, many 1-dim subsets of M not in a curve,
  - if $d = 2, n = 1, \epsilon = 0$, Lebesgue-null but full-dim subsets of a curve

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  - if $d = 2, n = 1, \epsilon = 0$, Lebesgue-null but full-dim subsets of a curve
- For $\epsilon = 0$ and $n = d$, there are Lebesgue-null subsets $E \subseteq M$ s.t.

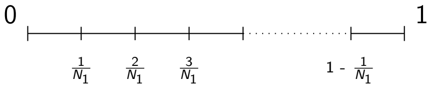
$$||\varphi_\lambda||_{L^p(E,\mu)} \quad \text{and} \quad ||\varphi_\lambda||_{L^p(M)}$$

obey the same bound for large p, up to a log loss.

Remarks on sharpness

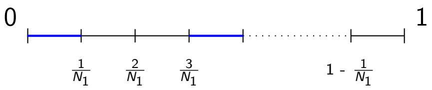
- (Work in progress) The exponent α_p of λ is sharp:
  - for $2n(1-\epsilon) < d-1$, for all $2 \leq p \leq \infty$.
  - for $2n(1-\epsilon) \geq d-1$ for the restricted range of large $p \geq p_0$.
- The source of the non-optimality is p_0, an artifact of the proof technique (more on this soon).
- The function $\Phi_p(\lambda)$ of sub-polynomial growth is explicit.

Random Cantor sets



Random Cantor sets

Step 1



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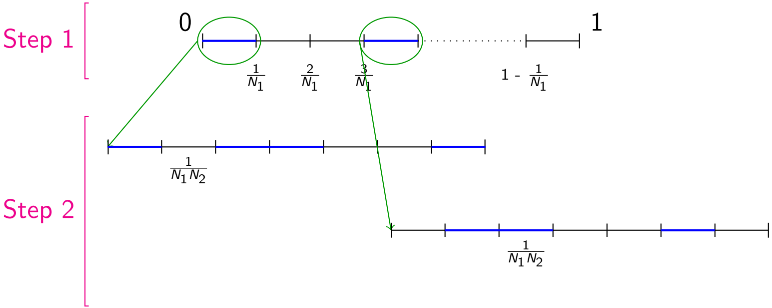

Parameters of the Cantor construction

Need

- a sequence of large constants N_k, say

$$N_k = N^k, \quad M_k = N_1 N_2 \cdots N_k.$$

- a sequence of small constants ϵ_k, say

$$\epsilon_k = \begin{cases} \frac{1}{k} & \text{if } \epsilon = 0, \\ \epsilon & \text{if } \epsilon > 0. \end{cases}$$

- For $k \geq 1$, choose a random binary sequence

$$\mathbf{Y}_k = \{Y_k(i_k) : 1 \leq i_k \leq N_k\}$$

whose entries are iid Bernoulli with

$$\mathbb{P}(Y_k(i_k) = 1) = p_k = N_k^{-\epsilon_k}.$$

Construction of the Cantor-type sets (ctd) : $d = 2, n = 1$

- The marker of selection

$$X_k(\mathbf{i}_k) = X_{k-1}(\mathbf{i}_{k-1}) Y_k(i_k), \quad \mathbf{i}_k = (i_1, \cdots, i_{k-1}).$$

- Define intervals

$$I_k(\mathbf{i}_k) = \alpha(\mathbf{i}_k) + \left[0, \frac{1}{M_k}\right], \quad \text{where}$$

$$\alpha(\mathbf{i}_k) = \frac{i_1 - 1}{N_1} + \frac{i_2 - 1}{N_1 N_2} + \cdots + \frac{i_k - 1}{N_1 N_2 \cdots N_k}.$$

- Start with $F_0 = [0, 1]^n$, and set

$$F_k = \bigcup \left\{ I_k(\mathbf{i}_k) : X_k(\mathbf{i}_k) = 1 \right\}, \qquad F = \bigcap_{k=1}^{\infty} F_k$$

- If $F \neq \emptyset$, lift $F \subseteq [0, 1]$ to $E \subseteq \Sigma$ via a coordinate chart.

The measure space of Cantor sets + The Cantor measures

$\mathbb{P}$: the product probability measure $\prod_{k=1}^{\infty} \mathbb{P}_k$, where $\mathbb{P}_k$ is the iid Bernoulli probability on $\mathbf{Y}_k$.

- For M_k and ϵ_k chosen as above,

$$\mathbb{P}(F \neq \emptyset) > 0.$$

$\mathbb{P}^*$: $\mathbb{P}$, conditional on the event that $F \neq \emptyset$:

$$\mathbb{P}^*(A) = \frac{\mathbb{P}^*(A \cap \{F \neq \emptyset\})}{\mathbb{P}(F \neq \emptyset)}.$$

- Equip every F with the natural Cantor measure μ:

$$\mu_k = \frac{1_{F_k}}{|F_k|}, \quad \mu_k \xrightarrow{*} \mu.$$

Plan

- Background and context
- Statement of the main results
- **Overview of the proof**
  - A review of an earlier proof
  - New features
  - Open questions

BGT-H proof for $d = 2, n = 1$: an overview

Step 1: Preparation

Step 2: The method of TT^*

Step3: Integration kernel estimates

Step 4: Young's convolution inequality

Step 1: Preparation of the operator

- Parametrix for a smooth, spectral projector
- A local representation of φ_λ
- Reduction to an oscillatory integral operator:

$$\mathscr{T}_\lambda f(x) = \int e^{i\lambda\psi(x,y)} a(x,y)\, f(y)\, dy, \quad x \in \mathbb{R}^2,$$

where $\psi(x,y) = -d_g(x,y)$. Now restrict $x = x(s) \in \Sigma$.

- A geodesic polar change of coordinates:

$$\mathscr{T}_\lambda f(x(s)) = \int_{c_1\epsilon}^{c_2\epsilon} \big(\mathscr{T}_\lambda^r f_r\big)(x(s))\, r\, dr, \text{ where}$$

$$\mathscr{T}_\lambda^r f(x(s)) = \int_{\mathbb{S}^1} e^{i\lambda\psi_r(x,\omega)} a_r(x(s),\omega) f(\omega)\, d\omega.$$

Step 2: The method of TT^*

- If we knew

$$||\mathscr{T}_\lambda^r f||_{L^p(\gamma)} \leq C\lambda^{\delta(p)-\frac{1}{2}}\Big(\int_{\mathbb{S}^1} |f(\omega)|^2\, d\omega\Big)^{\frac{1}{2}},$$

- then Minkowski $\implies$

$$||\mathscr{T}_\lambda f||_{L^p(\gamma)} \leq \int_{c_1\epsilon}^{c_2\epsilon} ||\mathscr{T}_\lambda^r f_r||_{L^p(\gamma)}\, dr$$

$$\leq C\lambda^{\delta(p)-\frac{1}{2}} \int_{c_1\epsilon}^{c_2\epsilon} ||f_r||_{L^2(\mathbb{S}^1)}\, dr \leq C\lambda^{\delta(p)-\frac{1}{2}}||f||_2.$$

- Thus aim to show

$$||\mathscr{T}_\lambda^r||^2_{L^2(\mathbb{S}^1)\to L^p(\gamma)} = ||\mathscr{T}_\lambda^r(\mathscr{T}_\lambda^r)^*||_{L^{p'}(\gamma)\to L^p(\gamma)} \leq C\lambda^{2\delta(p)-1}.$$

Step 3: Integration kernel estimates

- Write $\mathscr{T}_\lambda^r(\mathscr{T}_\lambda^r)^*$ as an integral operator,

$$T_\lambda f(x(t)) = \int_a^b K(t,s) f(x(s))\, ds, \text{ with}$$

- An explicit integration kernel

$$K(t,s) = \int_{\mathbb{S}^1} e^{i\lambda[\psi_r(x(t),\omega) - \psi_r(x(s),\omega)]}\, a_r(x(t),\omega)\, \overline{a}_r(x(s),\omega)\, d\omega.$$

- Method of stationary phase implies

$$|K(t,s)| \lesssim (1 + \lambda|t-s|)^{-\frac{1}{2}} = \tilde{K}_\lambda(t-s).$$

Summary

$\mathscr{T}_\lambda^r(\mathscr{T}_\lambda^r)^*$ is pointwise bounded by a convolution operator.

Step 4: Young's convolution inequality

For $1 \leq p_0, q_0, r_0 \leq \infty$,

$$||f * \tilde{K}_\lambda||_{r_0} \leq ||f||_{p_0} ||\tilde{K}_\lambda||_{q_0} \quad \text{provided} \quad \frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{r_0} + 1.$$

- Setting $p_0 = p'$, $r_0 = p$ and $q_0 = p/2$, get

$$||T_\lambda||_{L^{p'}(\gamma) \to L^p(\gamma)} \leq ||\tilde{K}_\lambda||_{L^{p/2}[0,1]}, \quad 2 \leq p \leq \infty.$$

- Since

$$\tilde{K}_\lambda(t) = (1 + \lambda|t|)^{-1/2},$$

its L^p-norms are easily computable.

What works for us, what doesn't

- BGT-H steps involving
  - Preparation of the spectral projection
  - The method of TT^*
  - Stationary phase on the integration kernel of TT^*

  go through with essentially no changes, but
- there is no Young's inequality for μ!

Proof distinctions: a generalized Young's inequality

- that does not use translation invariance of the underlying measure.

The replacement

Given

$$Tf(x) = \int K(x,y) f(y)\, d\mu(y)$$

and a choice of exponents $1 \le q, r, s \le \infty$ satisfying

$$\frac{1}{s} + \frac{1}{q} = \frac{1}{r} + 1,$$

we have

$$||Tf||_{L^r(\mu)} \le A_s^{1-\frac{s}{r}} B_s^{\frac{s}{r}} ||f||_{L^q(\mu)}, \text{ provided}$$

$$A_s := \sup_x \left[\int |K(x,y)|^s \, d\mu(y) \right]^{\frac{1}{s}}, \quad B_s := \sup_y \left[\int |K(x,y)|^s \, d\mu(x) \right]^{\frac{1}{s}}$$

are finite.

A fractal version of Young's inequality

- For us

$$A_s = B_s = \sup_u \int (1 + \lambda|u - v|)^{-\frac{s}{2}} d\mu(v),$$

where μ is the Cantor measure.

- A.s. upper bounds on A_s and B_s translate to operator bounds on T_λ.

- Estimation involves:
  - approximation of A_s and B_s using the absolutely continuous μ_k.
  - representing the approximation as a sum of partially deterministic components and centred random variables
  - large deviation inequalities, after suitable conditioning.

Ongoing work and concluding remarks

- Improvement of p_0:
  - need to harness the oscillation in $K(t, s)$
  - a stationary phase on random Cantor-type fractals.
- Improvement of α_p in special cases:
  - e.g. when E is a random subset of a curve $\gamma \subseteq M$ of nonvanishing geodesic curvature.
- What if E is deterministic and self-similar?
  - e.g. E is the Cantor middle third subset of a curve γ?

Thank you!